Estimation
~ Interval Estimation ~

區間估計

Interval Estimation

區間估計

區間估計

所謂區間估計（Interval Estimation）是以樣本估計母體時，所產生的估計值為一個區間的形式。
由於點估計可能無法準確推估母體參數，比較可行與客觀的方式就是利用抽樣出來的樣本，計算其統計量後，再給予一個可接受誤差範圍的區間，用於描述推估母體時，提供更多對於母體參數的正確、客觀訊息。

對稱型態

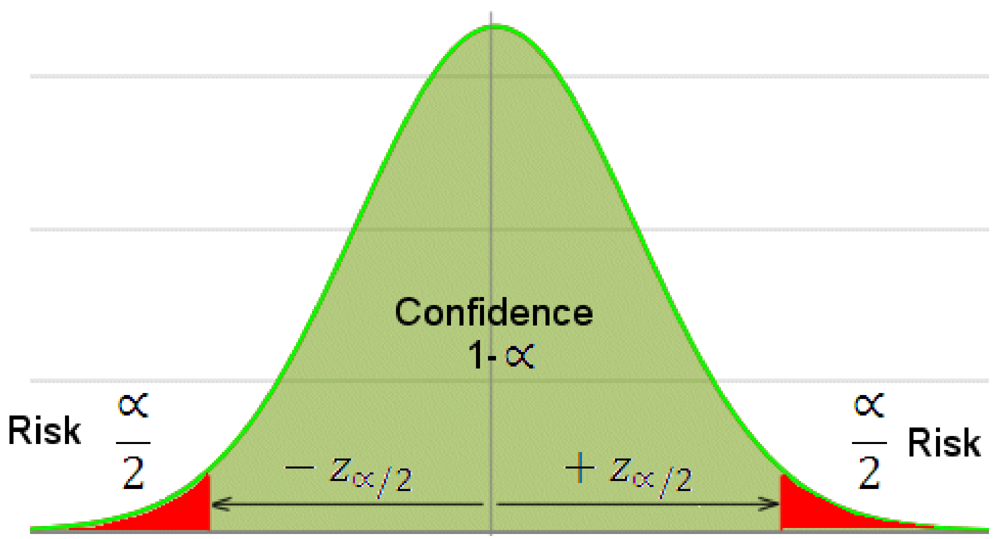
$$(\text{點估計值} - \text{誤差}) \leq \text{母體參數} \leq (\text{點估計值} + \text{誤差})$$

非對稱型態

$$\frac{\text{點估計值}}{\text{係數}_1} \leq \text{母體參數} \leq \frac{\text{點估計值}}{\text{係數}_2}$$

信賴水準

以樣本估計所形成的〝區間估計〞，主要是以母體參數在一個給定的「誤差」下所可能落點的區域。用以描述這個「誤差」的信心或可靠度，就是所謂的「信賴水準」（1-α），其中「α」稱為「顯著水準」（significance level）。

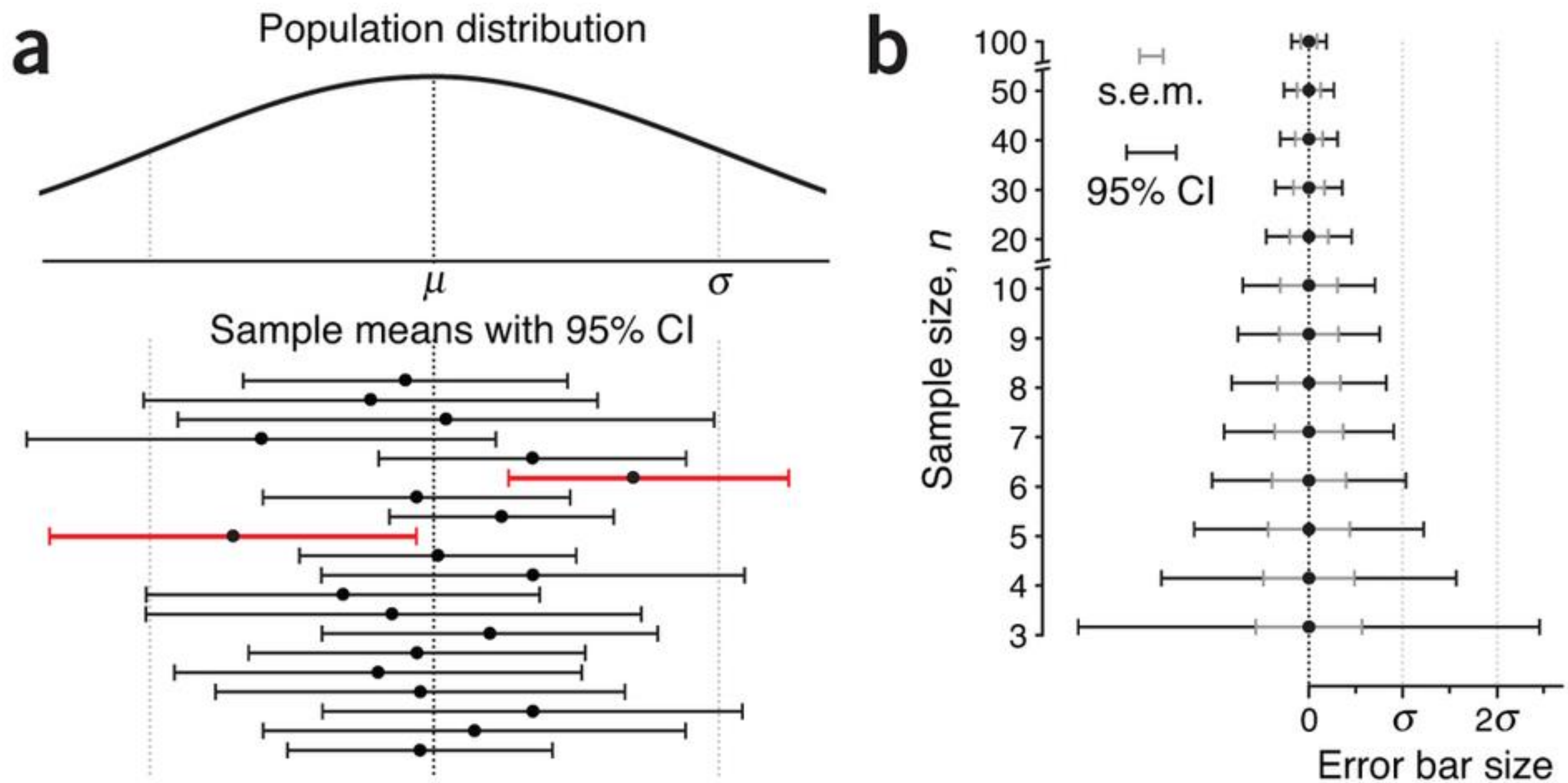


信賴水準（1-α）：

從一個母體不斷地反覆抽取n個樣本，並利用此n個樣本形成之區間，平均約有（1-α）的機會會包含此母體的參數。

信賴區間(confidence interval, CI)

s.e.m.：standard error of the mean
（樣本平均值的標準差）

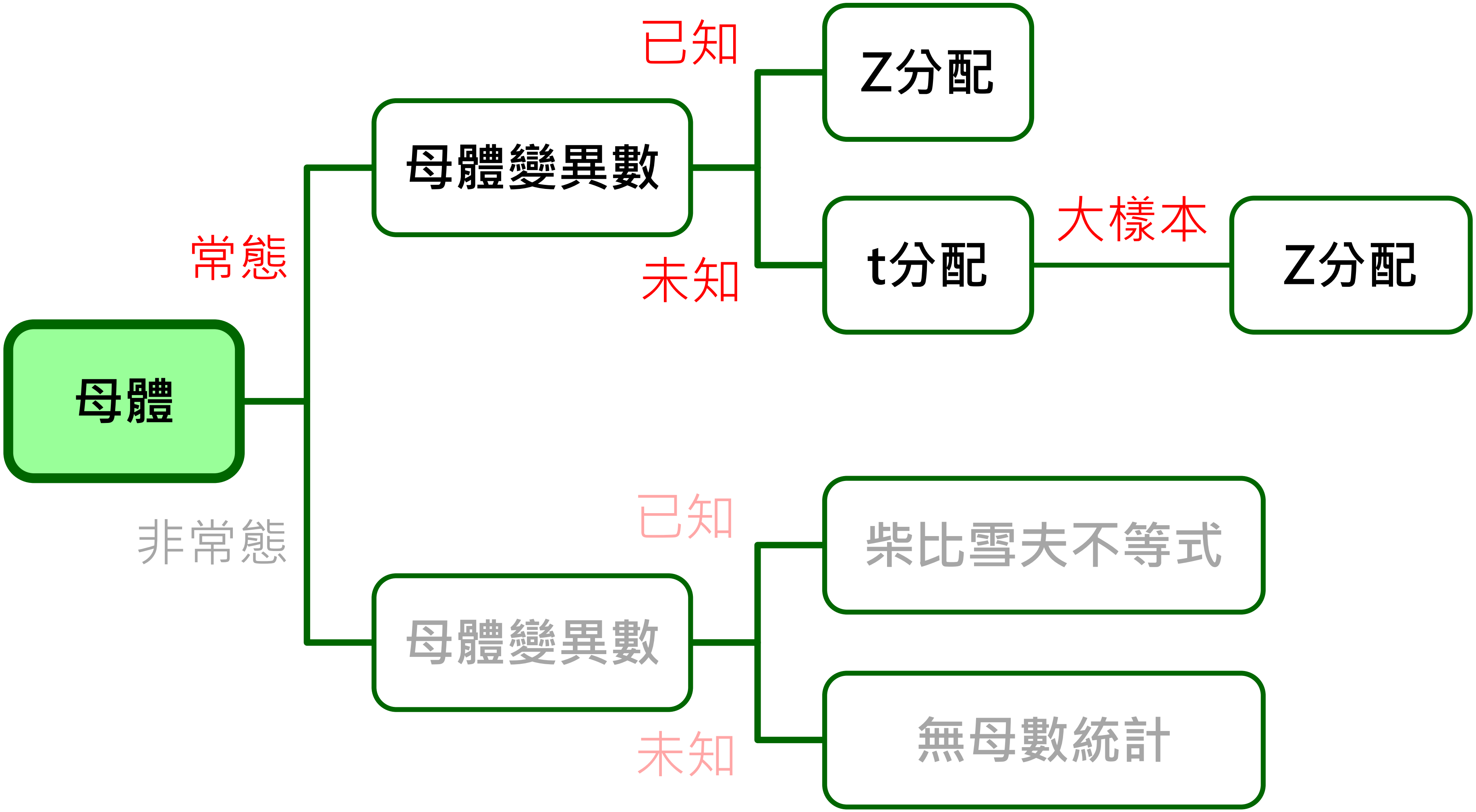


圖片來源：http://www.nature.com/nmeth/journal/v10/n10/fig_tab/nmeth.2659_F2.html

會想瞭解哪些統計量的信賴區間議題？

- 單母體
  - 母體平均數 — $\overline{x}$
  - 母體比例 — $\hat{p}$
  - 母體變異數 — s^2
- 兩母體
  - 兩母體平均差(獨立樣本) — $\overline{x}_1 - \overline{x}_2$
  - 兩母體平均差(成對樣本) — $\overline{x}_1 - \overline{x}_2$
  - 兩母體比例差 — $\hat{p}_1 - \hat{p}_2$
  - 兩母體變異數比 — s_1^2 / s_2^2

求得母體平均數信賴區間的方法



「統計學 二版」p13-9，李德治、林孟濡、童惠玲 著，博碩文化

Confidence Intervals for the Population Mean, μ

一個母體平均數的信賴區間

Confidence Interval Estimate

DCOVA

- An interval gives a range of values:
  - Takes into consideration variation in sample statistics from sample to sample
  - Based on observations from 1 sample
  - Gives information about closeness to unknown population parameters
  - Stated in terms of level of confidence
    - e.g. 95% confident, 99% confident
    - Can never be 100% confident

Confidence Interval Example

DCOVA

Cereal fill example

- Population has μ = 368 and σ = 15.
- If you take a sample of size n = 25 you know
  - $368 \pm 1.96 * 15 / \sqrt{25} = (362.12, 373.88)$ contains 95% of the sample means
  - When you don't know μ, you use $\bar{X}$ to estimate μ
    - If $\bar{X} = 362.3$ the interval is $362.3 \pm 1.96 * 15 / \sqrt{25} = (356.42, 368.18)$
    - Since $356.42 \leq \mu \leq 368.18$ the interval based on this sample makes a correct statement about μ.

But what about the intervals from other possible samples of size 25?

Confidence Interval Example

DCOVA

(continued)

Sample #	$\overline{X}$	Lower Limit	Upper Limit	Contain µ?
1	362.30	356.42	368.18	Yes
2	369.50	363.62	375.38	Yes
3	360.00	354.12	365.88	No
4	362.12	356.24	368.00	Yes
5	373.88	368.00	379.76	Yes

Confidence Interval Example

DCOVA

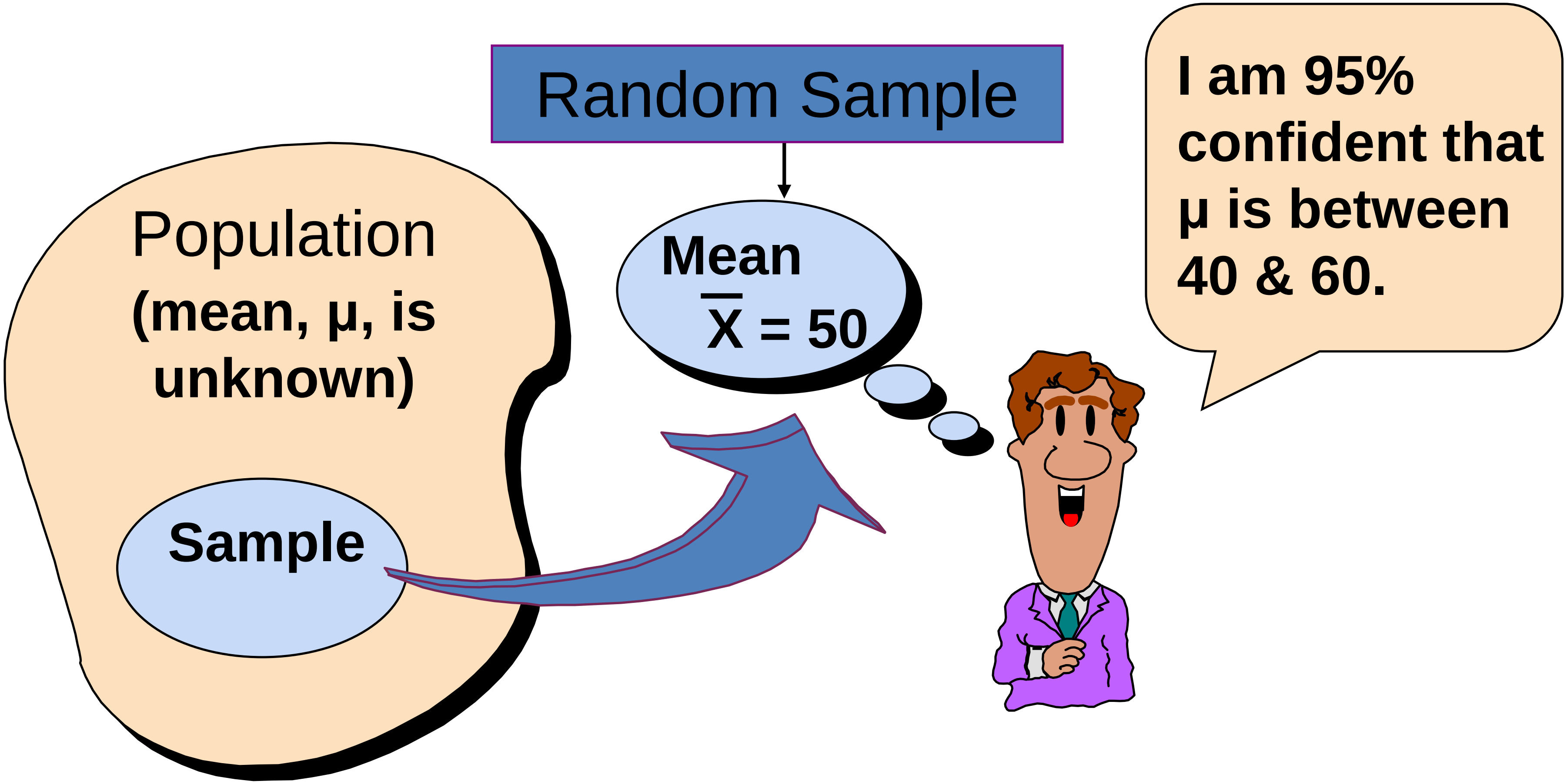
(continued)

- In practice you only take one sample of size n
- In practice you do not know μ so you do not know if the interval actually contains μ
- However you do know that 95% of the intervals formed in this manner will contain μ
- Thus, based on the one sample, you actually selected you can be 95% confident your interval will contain μ (this is a 95% confidence interval)

Note: 95% confidence is based on the fact that we used Z = 1.96.

Estimation Process

DCOVA



General Formula

DCOVA

- The general formula for all confidence intervals is:

Point Estimate $\pm$ (Critical Value)(Standard Error)

Where:

- Point Estimate is the sample statistic estimating the population parameter of interest
- Critical Value is a table value based on the sampling distribution of the point estimate and the desired confidence level
- Standard Error is the standard deviation of the point estimate

Confidence Level

DCOVA

- Confidence Level
  - Confidence the interval will contain the unknown population parameter
  - A percentage (less than 100%)

Confidence Level, $(1-\alpha)$

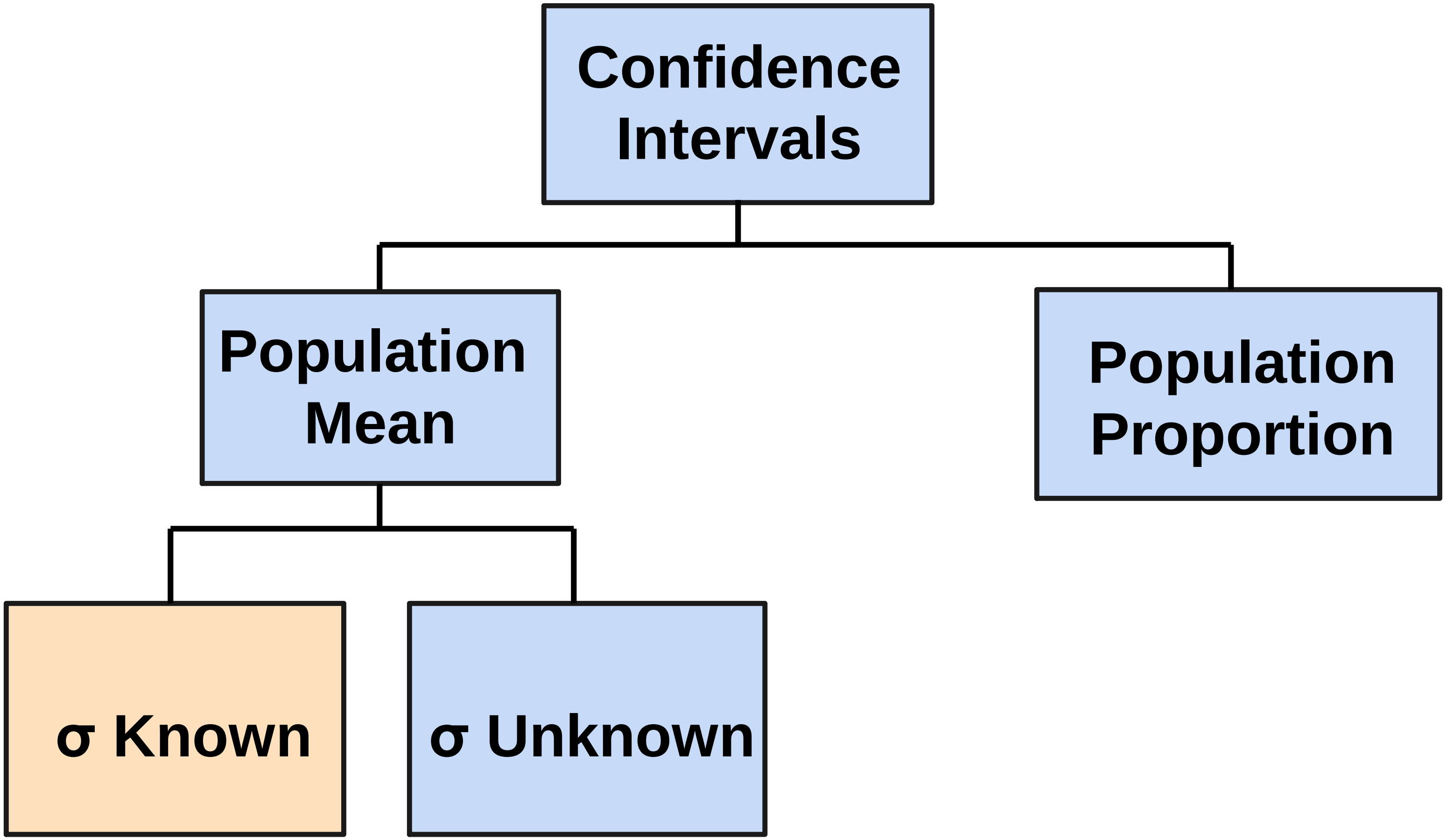
DCOVA

(continued)

- Suppose confidence level = 95%
- Also written $(1 - \alpha) = 0.95$, (so $\alpha = 0.05$)
- A relative frequency interpretation:
  - 95% of all the confidence intervals that can be constructed will contain the unknown true parameter
- A specific interval either will contain or will not contain the true parameter
  - No probability involved in a specific interval

Confidence Intervals

DCOVA



Confidence Interval for μ (σ Known)

DCOVA

- Assumptions
  - Population standard deviation σ is known
  - Population is normally distributed
  - If population is not normal, use large sample

- Confidence interval estimate:

$$\bar{X} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

請參閱

Sampling Distribution of the Mean

$$Z_{\bar{x}_n} = \frac{\bar{x}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} \longrightarrow N(0,1)$$

where $\bar{X}$ is the point estimate

$Z_{\alpha/2}$ is the normal distribution critical value for a probability of $\alpha/2$ in each tail

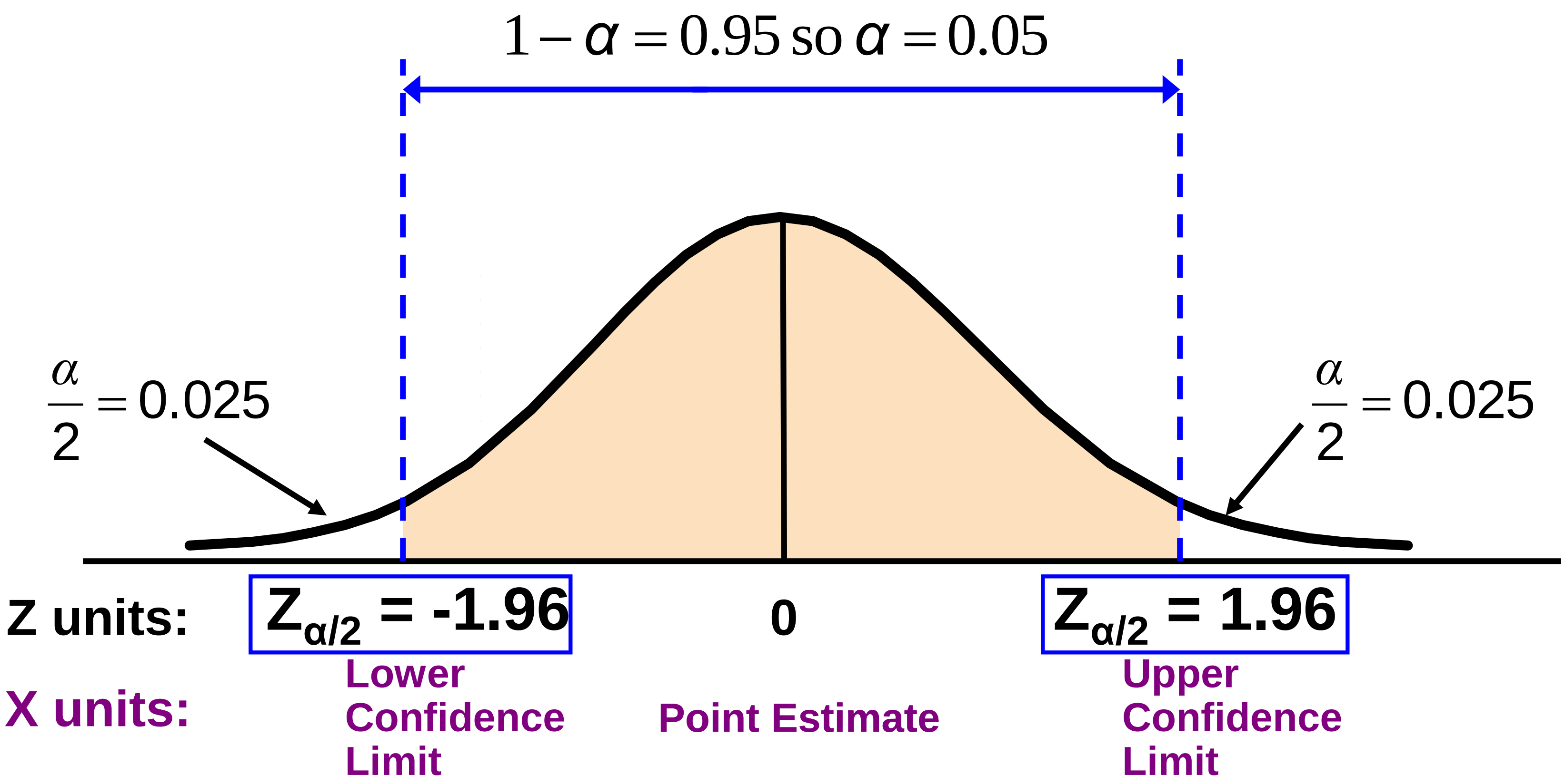
$\sigma/\sqrt{n}$ is the standard error

Finding the Critical Value, $Z_{\alpha/2}$

DCOVA

$$Z_{\alpha/2} = \pm 1.96$$

- Consider a 95% confidence interval:



Common Levels of Confidence

DCOVA

- Commonly used confidence levels are 90%, 95%, and 99%

Confidence Level	*Confidence Coefficient,* $1-\alpha$	$Z_{\alpha/2}$ *value*
80%	0.80	1.28
90%	0.90	1.645
95%	0.95	1.96
98%	0.98	2.33
99%	0.99	2.58
99.8%	0.998	3.08
99.9%	0.999	3.27

練習：

試著查表看

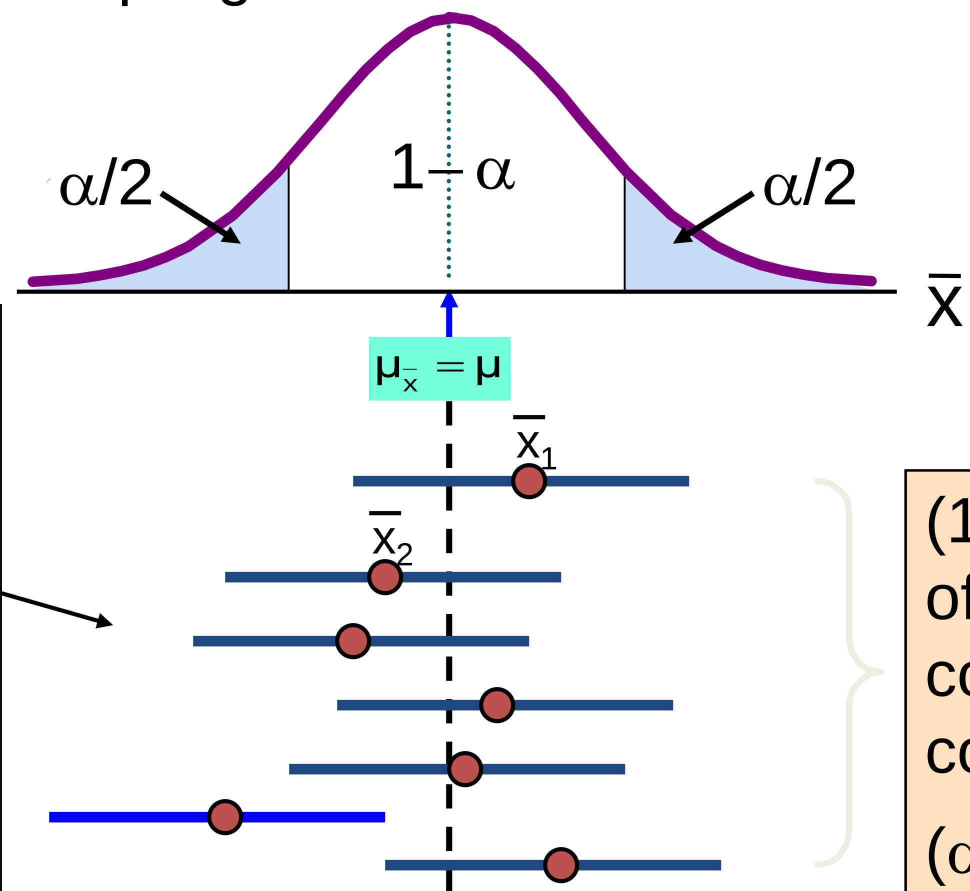
可不可以得

到相同結果？

Intervals and Level of Confidence

DCOVA

Sampling Distribution of the Mean



Confidence Intervals

Intervals extend from

$$\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

to

$$\bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

(1-α)x100% of intervals constructed contain μ;

(α)x100% do not.

案例

某銀行想要知道顧客的一般定期存款金額，以便業務拓展參考，於是隨機抽取49位客戶，得知這49位客戶平均一般定期存款金額為30萬。假設客戶一般定期存款金額為常態分配，變異數為64，試求平均一般定期存款金額的90%信賴區間？

「現代統計學」p147，吳柏林 著，五南圖書

案例解說

已知 $\bar{x} = 30$ ，n=49，$\sigma^2 = 64$ （σ已知）；
常態分配 → 分配對稱 → (1-α)=90% → α/2=0.05；

所以信賴區間為：

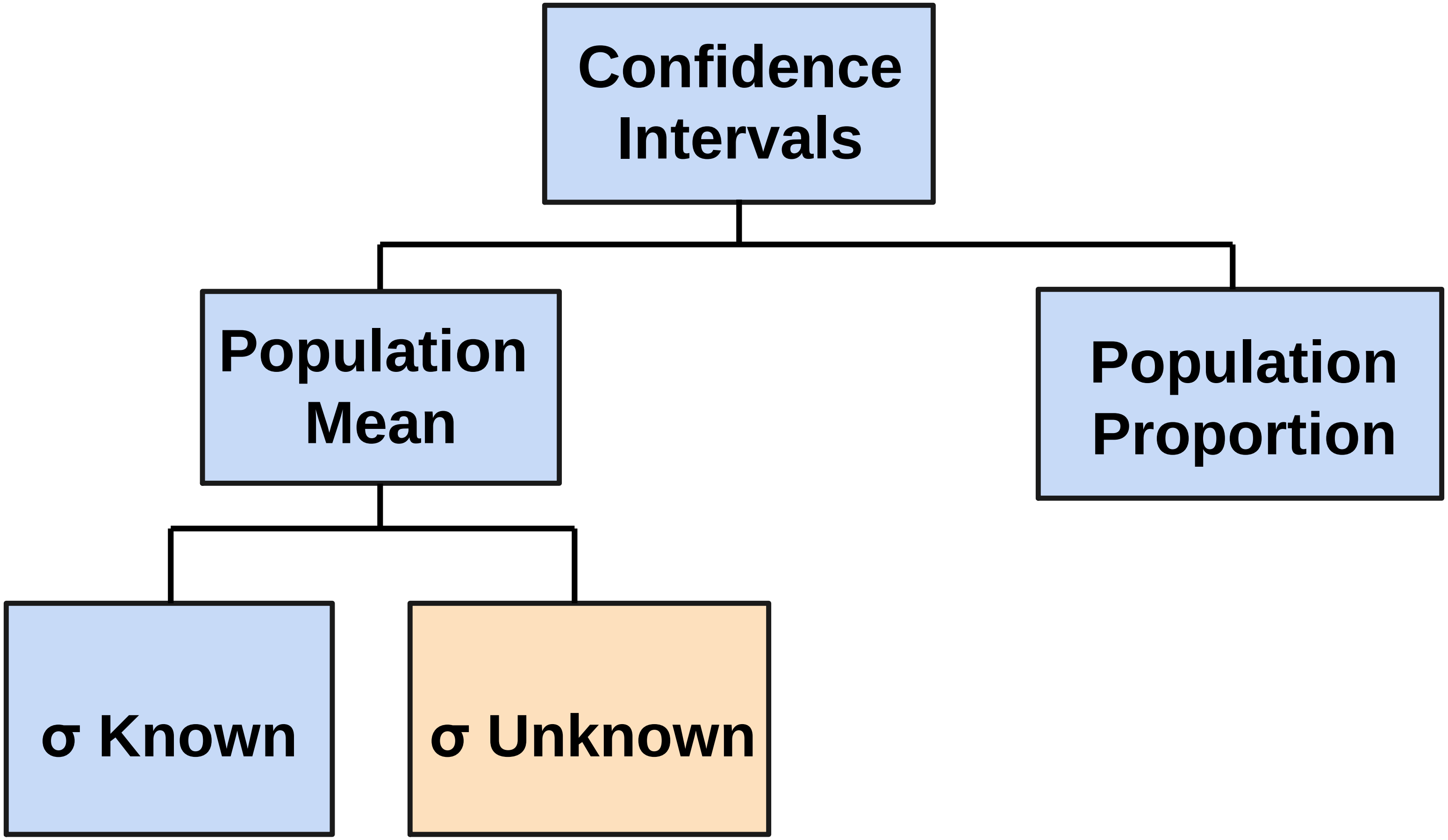
$$\left(\bar{x} - Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \quad \bar{x} + Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right) \Rightarrow$$

$$\left(\bar{x} - 1.645 \frac{8}{\sqrt{49}}, \quad \bar{x} + 1.645 \frac{8}{\sqrt{49}} \right)$$

→（ 30 ± 1.9 ）

Confidence Intervals

DCOVA



Do You Ever Truly Know σ?

- Probably not!

- In virtually all real world business situations, σ is not known.

- If there is a situation where σ is known then μ is also known (since to calculate σ you need to know μ.)

- If you truly know μ there would be no need to gather a sample to estimate it.

Confidence Interval for μ (σ Unknown)

DCOVA

- If the population standard deviation σ is unknown, we can substitute the sample standard deviation, S
- This introduces extra uncertainty, since S is variable from sample to sample
- So we use the t distribution instead of the normal distribution

Confidence Interval for μ (σ Unknown)

(continued)

DCOVA

- Assumptions
  - Population standard deviation is unknown
  - Population is normally distributed
  - If population is not normal, use large sample
- Use Student's t Distribution
- Confidence Interval Estimate:

$$\bar{X} \pm t_{\alpha/2} \frac{S}{\sqrt{n}}$$

請參閱

Student's t distribution

$$t = \frac{\bar{x}_n - \mu}{\sqrt{\frac{S^2}{n}}} \longrightarrow t(n-1)$$

(where $t_{\alpha/2}$ is the critical value of the t distribution with n -1 degrees of freedom and an area of α/2 in each tail)

Student’ s t Distribution

DCOVA

- The t is a family of distributions
- The $t_{\alpha/2}$ value depends on degrees of freedom (d.f.)
  - Number of observations that are free to vary after sample mean has been calculated

$$d.f. = n - 1$$

Degrees of Freedom (df)

DCOVA

Idea: Number of observations that are free to vary after sample mean has been calculated

Example: Suppose the mean of 3 numbers is 8.0

Let $X_1 = 7$

Let $X_2 = 8$

What is X_3?

If the mean of these three values is 8.0,
then X_3 must be 9
(i.e., X_3 is not free to vary)

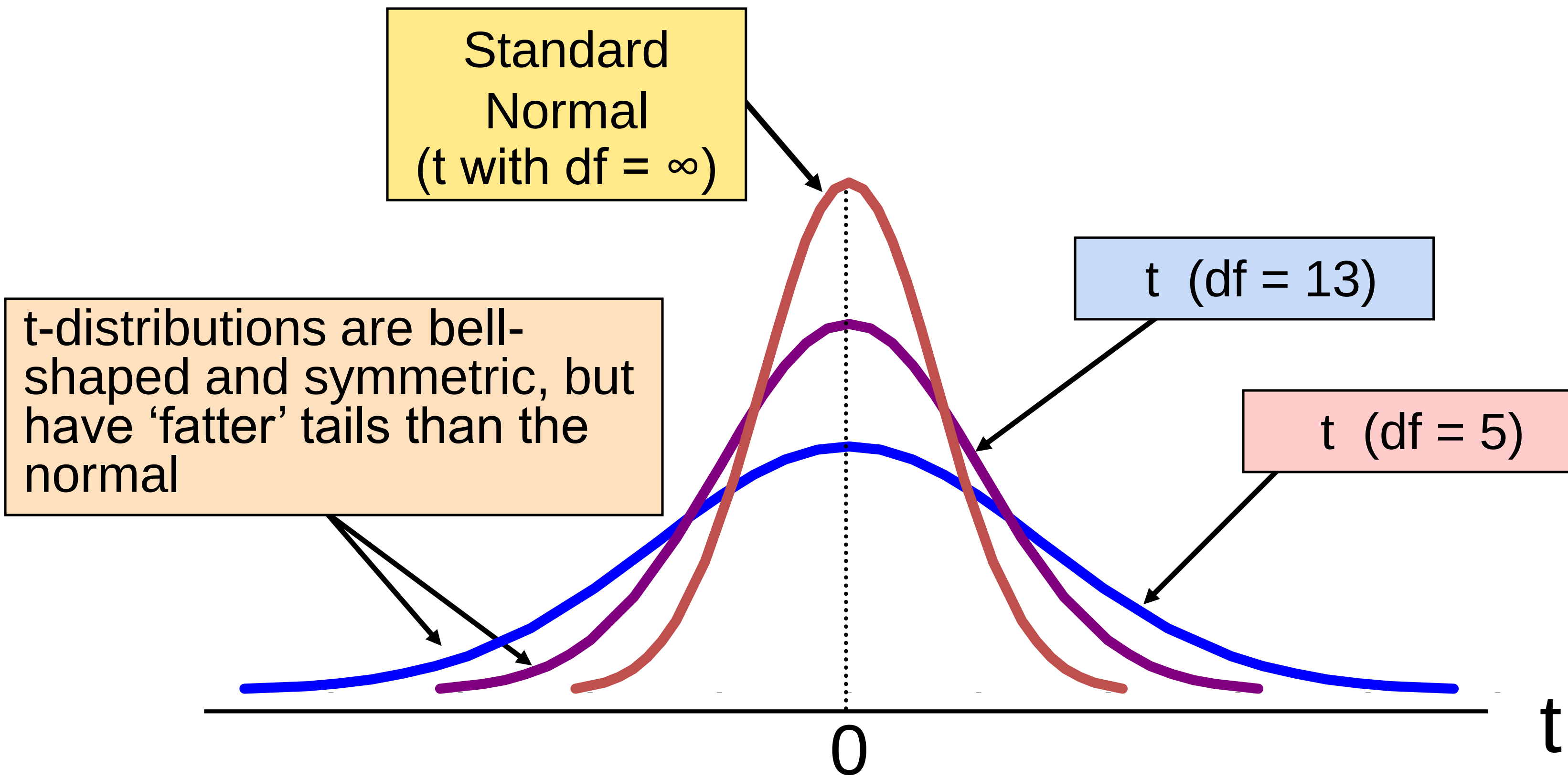
Here, n = 3, so degrees of freedom = n – 1 = 3 – 1 = 2

(2 values can be any numbers, but the third is not free to vary for a given mean)

Student's t Distribution

DCOVA

Note: t ⟶ Z as n increases



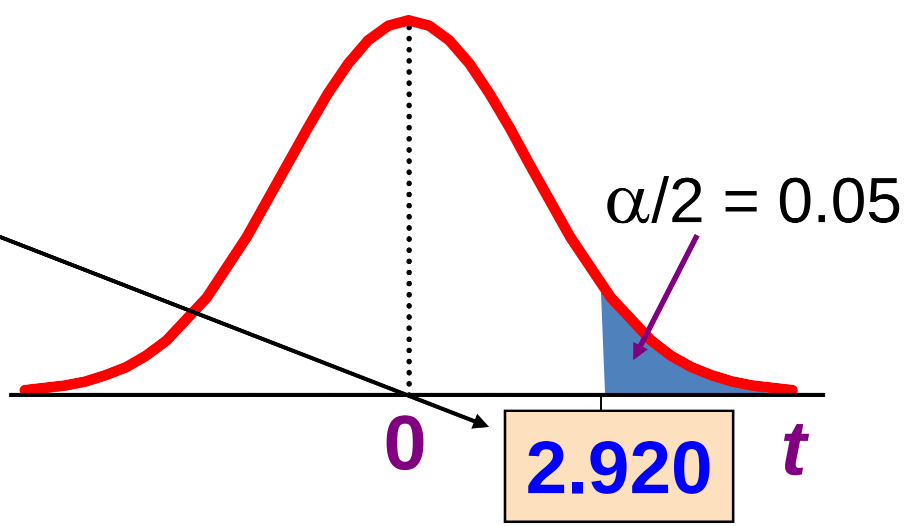
Student's t Table

DCOVA

	Upper Tail Area		
df	.10	**.05**	.025
1	3.078	6.314	12.706
2	1.886	**2.920**	4.303
3	1.638	2.353	3.182

The body of the table contains t values, not probabilities

Let: n = 3
df = *n* - 1 = 2
α = 0.10
$\alpha/2$ = 0.05



Selected t distribution values

DCOVA

With comparison to the Z value

Confidence Level	t (10 d.f.)	t (20 d.f.)	t (30 d.f.)	Z (∞ d.f.)
0.80	1.372	1.325	1.310	1.28
0.90	1.812	1.725	1.697	1.645
0.95	2.228	2.086	2.042	1.96
0.99	3.169	2.845	2.750	2.58

Note: $t \rightarrow Z$ as n increases

案例

台北市政府教育局想要估計該市小學生，平均每天收看電視所花費的時間。今隨機抽取26位小學生，得知這26位學童平均每天收看80分鐘的電視，樣本標準差為30分鐘？假設所調查的項目符合常態分配，試求該市小學生每天花費在看電視平均數 μ 的95%信賴區間？

「現代統計學」p148，吳柏林 著，五南圖書

案例解說

已知 $\bar{x}=80$ ，n=26，σ^2未知，但 s 可知為 30；
常態分配，σ未知 → 利用t分配 → 自由度df=25
(1-α)=95% → α/2=0.025；

所以信賴區間為：

$$\left(\bar{x}-t(n-1)\frac{s}{\sqrt{n}},\qquad \bar{x}+t(n-1)\frac{\sigma}{\sqrt{n}}\right)$$

$$\Rightarrow\left(\bar{x}-t_{0.025}(25)\frac{30}{\sqrt{26}},\qquad \bar{x}+t_{0.025}(25)\frac{30}{\sqrt{26}}\right)$$

$$\Rightarrow\left(80-2.06\frac{30}{\sqrt{26}},\qquad 80+2.06\frac{30}{\sqrt{26}}\right)\ \rightarrow\ (\ 80\pm 12\ )$$

Confidence Intervals for the Population Proportion, π

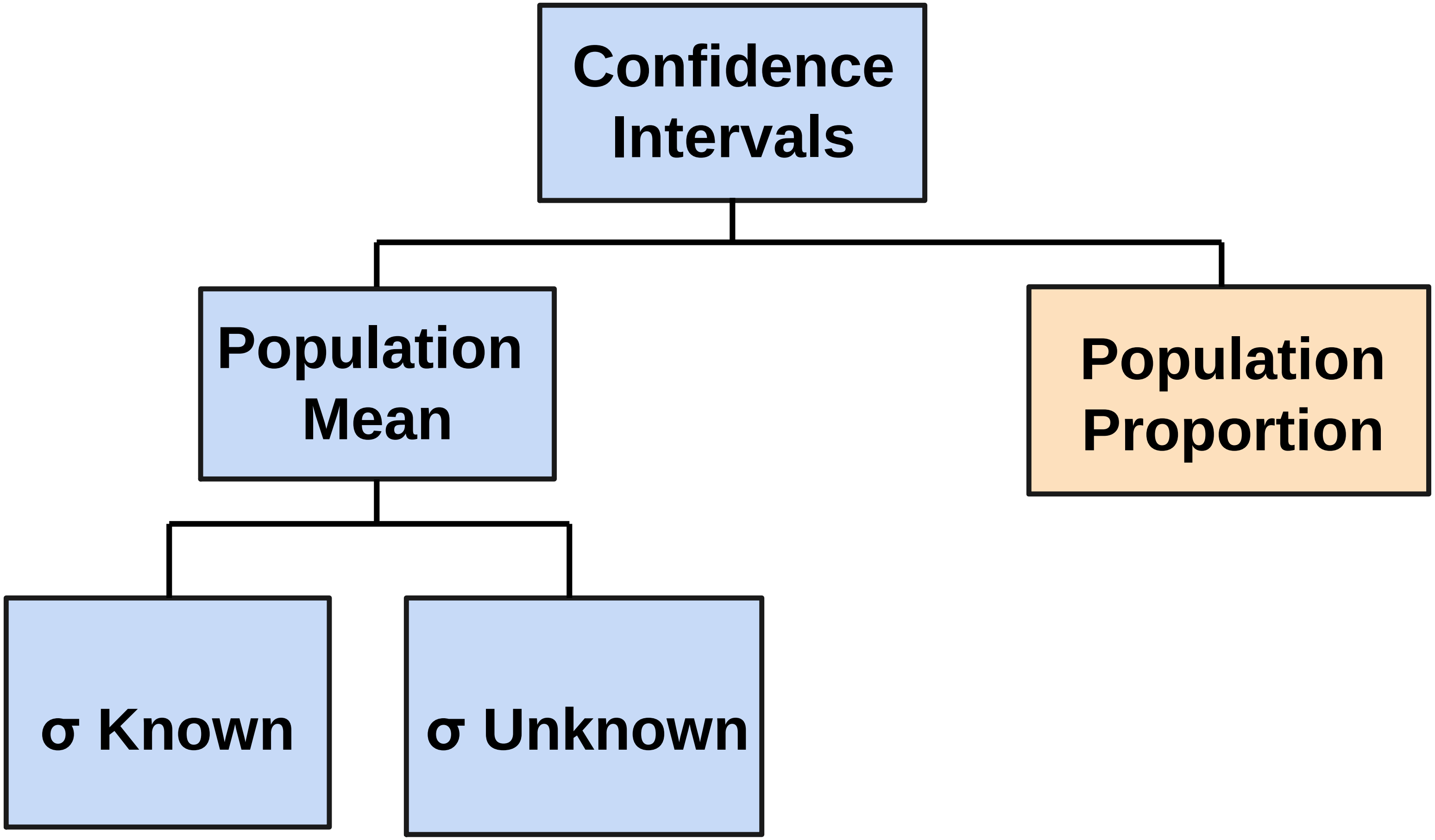
一個母體比例的信賴區間

對照

比例符號	母體	比例統計量
課本	π	p
老師習慣	p	$\hat{p}$

Confidence Intervals

DCOVA



Confidence Intervals for the Population Proportion, π

DCOVA

- An interval estimate for the population proportion (π) can be calculated by adding an allowance for uncertainty to the sample proportion (p)

對照

比例符號	母體	比例統計量
課本	π	p
老師習慣	p	$\hat{p}$

Confidence Intervals for the Population Proportion, π

(continued)

DCOVA

- Recall that the distribution of the sample proportion is approximately normal if the sample size is large, with standard deviation

$$\sigma_p = \sqrt{\frac{\pi(1-\pi)}{n}}$$

- We will estimate this with sample data:

$$\sqrt{\frac{p(1-p)}{n}}$$

Confidence Interval Endpoints

DCOVA

- Upper and lower confidence limits for the population proportion are calculated with the formula

請參閱

$$p \pm Z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

Sampling Distribution of the Proportion

$$Z = \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

請對照老師的比例符號

- where
  - $Z_{\alpha/2}$ is the standard normal value for the level of confidence desired
  - p is the sample proportion
  - n is the sample size
- Note: must have np > 5 and n(1-p) > 5

案例

衛生署想要調查全國大專院校學生抽煙人口比例，於是隨機收取100位大專生，發現有19位是抽煙人口，試求抽煙人口比例之95%信賴區間？

「現代統計學」p155，吳柏林 著，五南圖書

案例解說

請對照老師的比例符號

求「比例」→ 想到 $\hat{p}$ → n=100，k=19；
調查為人口數、題目未特別交代
→利用常態分配解題 → (1-α)=95% → α/2=0.025；

所以信賴區間為：

$$\left(\hat{p}-Z_{\frac{\alpha}{2}}\sqrt{\frac{(\hat{p}(1-\hat{p}))}{n}},\quad \hat{p}+Z_{\frac{\alpha}{2}}\sqrt{\frac{(\hat{p}(1-\hat{p}))}{n}}\right)\Rightarrow$$

$$\left(\frac{19}{100}-1.96\sqrt{\frac{(0.19)(0.81)}{100}},\quad \frac{19}{100}+1.96\sqrt{\frac{(0.19)(0.81)}{100}}\right)\Rightarrow$$

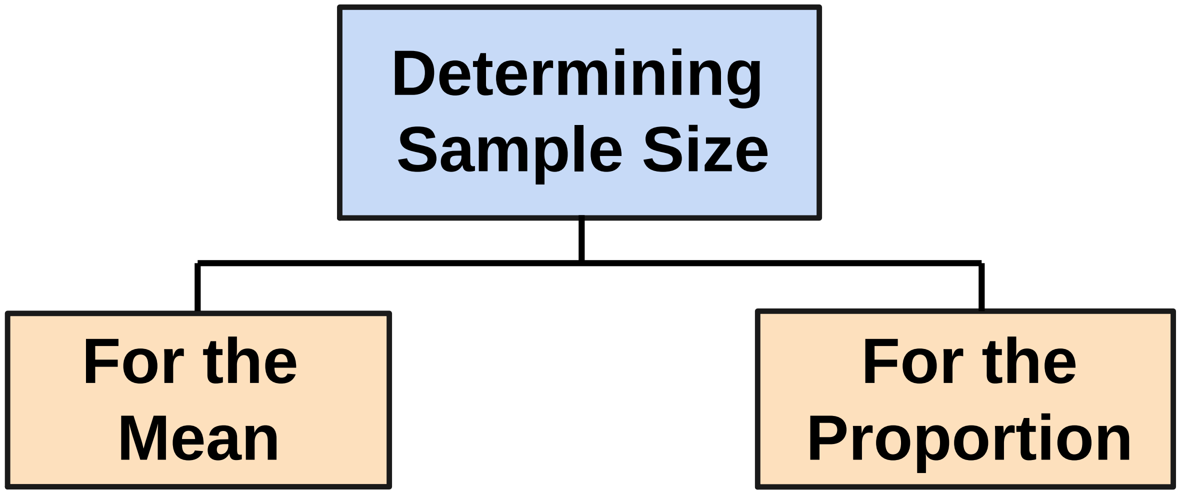
→ (0.19 ± 0.08)

Required Sample Size

符合信賴區間要求的樣本數

Determining Sample Size

DCOVA



Sampling Error

DCOVA

- The required sample size can be found to reach a desired margin of error (e) with a specified level of confidence $(1 - \alpha)$

- The margin of error is also called sampling error
  - the amount of imprecision in the estimate of the population parameter
  - the amount added and subtracted to the point estimate to form the confidence interval

Determining Sample Size

DCOVA

Determining Sample Size

For the Mean

$$\bar{X} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Sampling error (margin of error)

$$e = Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Determining Sample Size

DCOVA

(continued)

Determining Sample Size

For the Mean

$$e = Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Now solve for n to get

$$n = \frac{Z_{\alpha/2}^2 \sigma^2}{e^2}$$

Determining Sample Size

DCOVA

(continued)

- To determine the required sample size for the mean, you must know:
  - The desired level of confidence (1 - α), which determines the critical value, $Z_{\alpha/2}$
  - The acceptable sampling error, e
  - The standard deviation, σ

Required Sample Size Example

DCOVA

If σ = 45, what sample size is needed to estimate the mean within ± 5 with 90% confidence?

$$n = \frac{Z^2 \sigma^2}{e^2} = \frac{(1.645)^2 (45)^2}{5^2} = 219.19$$

So the required sample size is **n = 220**

(Always round up)

If σ is unknown

DCOVA

- If unknown, σ can be estimated when using the required sample size formula
  - Use a value for σ that is expected to be at least as large as the true σ
  - Select a pilot sample and estimate σ with the sample standard deviation, S

Determining Sample Size

(continued)

DCOVA

Determining Sample Size

For the Proportion

$$e = Z\sqrt{\frac{\pi(1-\pi)}{n}}$$

Now solve for n to get

$$n = \frac{Z_{\alpha/2}^{2}\,\pi(1-\pi)}{e^{2}}$$

Determining Sample Size

(continued)

DCOVA

- To determine the required sample size for the proportion, you must know:
  - The desired level of confidence (1 - α), which determines the critical value, $Z_{\alpha/2}$
  - The acceptable sampling error, e
  - The true proportion of events of interest, π
    - π can be estimated with a pilot sample if necessary (or conservatively use 0.5 as an estimate of π)

Required Sample Size Example

DCOVA

How large a sample would be necessary to estimate the true proportion defective in a large population within ±3%, with 95% confidence?

(Assume a pilot sample yields $p = 0.12$)

Required Sample Size Example

(continued)

DCOVA

Solution:

For 95% confidence, use $Z_{\alpha/2} = 1.96$

e = 0.03

p = 0.12, so use this to estimate π

$$n = \frac{Z_{\alpha/2}^2 \pi(1-\pi)}{e^2} = \frac{(1.96)^2(0.12)(1-0.12)}{(0.03)^2} = 450.74$$

So use n = 451

案例1

已知某大學有8000位學生，根據過去的一項調查，發生這些學生每月平均零用錢為8200元，標準差為1200元。現欲進行抽樣調查，要求誤差在95%的信賴水準下，不超過 ± 5%的時候，至少應抽選多少樣本數？

「應用統計學 二版」p237，李德治、童惠玲 著，博碩文化

案例2

某校依照過去經驗知，大約有10%的學生反對學費上漲，現欲抽樣以確認反對學費上漲的比例，試問希望誤差在5%以內的機率為0.99時，至少應抽取幾個樣本？

「應用統計學 二版」p241，李德治、童惠玲 著，博碩文化

The End

案例1：至少取樣33個

案例2：至少取樣238個